

Quasi-Exactly Solvable Deformations of Gaudin Models and “Quasi-Gaudin Algebras”.

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Abstract

A new class of completely integrable models is constructed. These models are deformations of the famous integrable and exactly solvable Gaudin models. In contrast with the latter, they are quasi-exactly solvable, i.e. admit the algebraic Bethe ansatz solution only for some limited parts of the spectrum. An underlying algebra responsible for both the phenomena of complete integrability and quasi-exact solvability is constructed. We call it “quasi-Gaudin algebra” and demonstrate that it is a special non-Lie-algebraic deformation of the ordinary Gaudin algebra.

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1 Introduction

A quantum model is usually called *exactly solvable* if all solutions of its spectral problem can be found algebraically. Recently a new important class of the so-called *quasi-exactly solvable* models has been discovered [1, 2, 3, 4]. These models are distinguished by the fact that only certain limited parts of their spectra admit an algebraic construction. A detailed exposition of different theories explaining the phenomenon of quasi-exact solvability and proposing constructive methods for building and solving such models can be found in papers [5, 6, 7, 9, 10, 8, 11] and in the book [12].

The quasi-exactly solvable models can be considered as *deformations* of exactly solvable ones [12]. Usually, the deformation of a given exactly solvable model leads to an infinite sequence of quasi-exactly solvable models with different hamiltonians and different number of exact solutions [12].

For example, the simplest exactly solvable model of a multi-dimensional harmonic oscillator with hamiltonian

$$H = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^d b_i^2 x_i^2 \quad (1.1)$$

can be deformed into an infinite sequence of quasi-exactly solvable models of multi-dimensional sextic anharmonic oscillators with hamiltonians

$$H_n = -\sum_{i=1}^d \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^d b_i^2 x_i^2 + 2ar^2 \left[\sum_{i=1}^d b_i x_i^2 - 2n - 1 - \frac{d}{2} \right] + a^2 r^6, \quad n = 0, 1, \dots \quad (1.2)$$

In contrast with (1.1), each of models (1.2) has only $(n+d)!/(n!d!)$ algebraically constructable solutions [3]. Here the role of a deformation parameter is played by a .

It is worth stressing that both models (1.1) and (1.2) are completely integrable in the sense that they have enough number of mutually commuting integrals of motion (see e.g. [12]). Thus, the above example clearly demonstrates that the deformation procedure may preserve the integrability property. There are many other examples of such a sort which can be found in ref. [12].

It is natural to assume that any completely integrable and exactly solvable quantum model can be deformed into an infinite set of integrable and quasi-exactly solvable models. If this assumption is true, then the following interesting problem immediately arises.

The Problem. It is well known that most of physically interesting *integrable* and *exactly solvable* models can be obtained and solved in the framework of the celebrated *R*-matrix approach [13, 14, 15]. A mathematical structure lying in the ground of this approach and responsible for both the phenomena of integrability and exact solvability is the so-called Yang – Baxter (YB) algebra (see e.g. the book [16]). Using its generators, one can easily construct not only all the integrals of motion of a model (and thus ensure its integrability) but also build all eigenvectors and eigenvalues of the corresponding spectral problem (and thus ensure its exact solvability in the framework of the so-called Algebraic Bethe Ansatz [15]). Let us now consider any quasi-exactly solvable deformation of such a model that preserves its integrability property. If such a deformation does exist, then it is natural to ask ourselves if there is any algebra responsible for both the phenomena of *integrability* and *quasi-exact solvability* of the deformed model? In other words, is there any analogue of the YB algebra (some kind of "quasi-YB algebra") whose generators could be used not only for constructing

the integrals of motion of the arising quasi-exactly solvable models but also for building their algebraically constructable eigenvalues and eigenvectors? We mean here some analogue of the Algebraic Bethe Ansatz. And the last and, may be, the most important question: is it possible to interpret this quasi-YB algebra as a *deformation* of the ordinary YB algebra?

It is obvious that if solution of this problem does actually exist and the quasi-YB algebras will be found, then, developing a regular method for building their representations we obtain a big factory of new integrable and quasi-exactly solvable models. Since the ordinary YB algebras give rise to many physically interesting exactly solvable systems of quantum mechanics, statistical physics and field theory, it would be natural to expect that the set of quasi-exactly solvable models associated with quasi-YB algebras also will contain a lot of physically interesting systems.

It seems that the posed problem has a positive solution. In this paper we consider the simplest case of the so-called $sl(2)$ -Gaudin algebra $G[sl(2)]$ which is closely related to the classical YB algebra and leads to the completely integrable Gaudin models solvable by means of Bethe ansatz [17, 18] (section 2). In sections 3 and 4 we construct a special non-Lie algebraic deformation of this algebra which automatically produces an infinite series of integrable models. We call this algebra the "quasi- $G[sl(2)]$ algebra" and the corresponding models – the "quasi-Gaudin models". In sections 5, 6 and 7 we show that all these models are quasi-exactly solvable in the sense that they admit only a partial Bethe ansatz solution of spectral problem. In section 8 we construct a simple realization of quasi- $G[sl(2)]$ algebra and later, in section 9, demonstrate in an independent way that quasi-Gaudin models associated with this concrete realization are actually quasi-exactly solvable. In the last section 10 we introduce a concept of quasi- $sl(2)$ algebra which can be viewed as a special limiting case of quasi- $G[sl(2)]$.

2 The Gaudin algebra

In this section we remind the reader some basic facts concerning Gaudin algebras, their representations and properties of the associated Gaudin models. More detailed exposition of this subject can be found in refs. [17, 18, 8, 12].

The Gaudin algebra $G[sl(2)]$ is an infinite-dimensional extension of the ordinary $sl(2)$ algebra. Its three generators, $\tilde{S}(\lambda) = \{S^-(\lambda), S^0(\lambda), S^+(\lambda)\}$ with $\lambda \in C$ obey the commutation relations

$$\begin{aligned} S^0(\lambda)S^0(\mu) - S^0(\mu)S^0(\lambda) &= 0, \\ S^\pm(\lambda)S^\pm(\mu) - S^\pm(\mu)S^\pm(\lambda) &= 0, \\ S^0(\lambda)S^\pm(\mu) - S^\pm(\mu)S^0(\lambda) &= \pm \frac{S^\pm(\lambda) - S^\pm(\mu)}{\mu - \lambda}, \\ S^-(\lambda)S^+(\mu) - S^+(\mu)S^-(\lambda) &= 2 \frac{S^0(\lambda) - S^0(\mu)}{\mu - \lambda}, \end{aligned} \tag{2.1}$$

generalizing those of $sl(2)$. Using (2.1), one can easily prove that the operators

$$C(\lambda) = S^0(\lambda)S^0(\lambda) - \frac{1}{2}S^-(\lambda)S^+(\lambda) - \frac{1}{2}S^+(\lambda)S^-(\lambda) \tag{2.2}$$

form a commutative family,

$$[C(\lambda), C(\mu)] = 0, \quad (2.3)$$

and thus, can be interpreted as integrals of motion of a certain quantum completely integrable model. The latter is known under name of Gaudin model.

The role of the “Hilbert space” in which the operators $C(\lambda)$ act is played by the representation space of Gaudin algebra. In order to construct it one needs to fix the lowest weight vector $|0\rangle$ and the lowest weight function $f(\lambda)$ obeying the relations

$$S^0(\lambda)|0\rangle = f(\lambda)|0\rangle, \quad S^-(\lambda)|0\rangle = 0. \quad (2.4)$$

After this, we can define the representation space as a linear hull of vectors

$$|\xi_1, \dots, \xi_m\rangle = S^+(\xi_m) \dots S^+(\xi_1)|0\rangle \quad (2.5)$$

with arbitrary m and ξ_1, \dots, ξ_m . We shall denote this representation space by $W_{f(\lambda)}$.

The “Schrödinger equation” for the Gaudin model reads now

$$C(\lambda)\phi = E(\lambda)\phi, \quad \phi \in W_{f(\lambda)}. \quad (2.6)$$

The beauty of this equation lies in the fact that all its solutions can be obtained algebraically. For example, using commutation relations (2.1) and formulas (2.4) it is easy to check that the lowest weight vector is always a solution of the Gaudin problem:

$$C(\lambda)|0\rangle = \left(f^2(\lambda) + \frac{\partial}{\partial \lambda} f(\lambda) \right) |0\rangle. \quad (2.7)$$

The remaining solutions of equation (2.6) can be obtained by means of the so-called algebraic Bethe ansatz

$$\phi = S^+(\xi_m)S^+(\xi_{m-1}) \dots S^+(\xi_2)S^+(\xi_1)|0\rangle, \quad (2.8)$$

in which m is an arbitrary non-negative integer and ξ_1, \dots, ξ_m are some unknown numbers. It can be demonstrated that vector (2.8) is an eigenvector of the Gaudin operator with the eigenvalue

$$E(\lambda) = f^2(\lambda) + \frac{\partial}{\partial \lambda} f(\lambda) + \sum_{i=1}^m \frac{f(\lambda) - f(\xi_i)}{\lambda - \xi_i} \quad (2.9)$$

if the following conditions hold

$$\sum_{k=1, k \neq i}^m \frac{1}{\xi_i - \xi_k} + f(\xi_i) = 0, \quad i = 1, \dots, m. \quad (2.10)$$

These conditions are known as Bethe ansatz equations. It is known that the constructed solutions with $m = 0, 1, \dots$ expire all possible solutions of the Gaudin spectral problem and therefore the latter is *exactly solvable*.

3 The quasi-Gaudin algebra

In this section we consider a special modification of Gaudin algebra which also leads to completely integrable quantum systems.

Let $\tilde{S}_n(\lambda) = \{S_n^-(\lambda), S_n^0(\lambda), S_n^+(\lambda)\}$, $n \in Z$, $\lambda \in C$ denote the operators obeying the relations

$$\begin{aligned} S_n^0(\lambda)S_n^0(\mu) - S_n^0(\mu)S_n^0(\lambda) &= 0, \\ S_{n\pm 1}^\pm(\lambda)S_n^\pm(\mu) - S_n^\pm(\mu)S_{n\pm 1}^\pm(\lambda) &= 0, \\ S_{n\pm 1}^0(\lambda)S_n^\pm(\mu) - S_n^\pm(\mu)S_{n\pm 1}^0(\lambda) &= \pm \frac{S_n^\pm(\lambda) - S_n^\pm(\mu)}{\mu - \lambda}, \\ S_{n+1}^-(\lambda)S_n^+(\mu) - S_n^+(\mu)S_{n+1}^-(\lambda) &= 2 \frac{S_n^0(\lambda) - S_n^0(\mu)}{\mu - \lambda}. \end{aligned} \quad (3.1)$$

We consider (3.1) as the defining relations of a certain infinite-dimensional algebra. It is not difficult to see that this algebra is very similar to the Gaudin algebra but, in contrast with the latter, it *is not* a Lie algebra². We call it the “quasi-Gaudin algebra” or, more concretely, “quasi- $G[sl(2)]$ ”.

The algebra (3.1) has many remarkable properties. First of all, only from the “quasi-commutation relations” (3.1) it immediately follows that the operator valued functions

$$C_n(\lambda) = S_n^0(\lambda)S_n^0(\lambda) - \frac{1}{2}S_{n+1}^-(\lambda)S_n^+(\lambda) - \frac{1}{2}S_{n-1}^+(\lambda)S_n^-(\lambda) \quad (3.2)$$

form commutative families for any n :

$$[C_n(\lambda), C_n(\mu)] = 0. \quad (3.3)$$

These functions are obvious generalizations of the Gaudin integrals of motion. Note, however, that the commutators between $C_n(\lambda)$ and $C_m(\lambda)$ do not vanish if $m \neq n$.

The property (3.3) suggests to interpret $C_n(\lambda)$ as generating functions of commuting integrals of motion for certain completely integrable quantum models. In fact, (3.3) describes an infinite sequence of such models, since the integer n can be chosen arbitrarily. This is the first important difference with the Gaudin case when an analogous construction leads to a single completely integrable model — the Gaudin model.

Exactly as in the case of Gaudin algebra, the construction of these models needs the specification of the “Hilbert space”. It would be natural to identify it with the representation space of our algebra. The latter can be constructed in the same way as in the Gaudin case. First of all, we need the lowest weight vector $|0\rangle$ which, as in the Gaudin case, should be an eigenvector of both the lowering and neutral operators $S_n^-(\lambda)$ and $S_n^0(\lambda)$. The most general relations expressing this fact and compatible with the quasi-commutation relations (3.1) read

$$S_n^0(\lambda)|0\rangle = (F(\lambda) + nG(\lambda))|0\rangle, \quad S_n^-(\lambda)|0\rangle = nG(\lambda)|0\rangle, \quad (3.4)$$

²In section 8 we demonstrate that (3.1) can be viewed as a deformation of the Gaudin algebra.

where $F(\lambda)$ and $G(\lambda)$ are certain arbitrarily fixed functions³. Note that the lowest weight vector $|0\rangle$ is not annihilated by the “lowering operators” $S_n^-(\lambda)$ except for the case of $n = 0$. This fact, which will play a determining role in our further considerations, gives us the second important difference with the Gaudin case.

After fixing the “lowest weight functions” $F(\lambda)$ and $G(\lambda)$, we can define the representation space as a linear hull of vectors

$$|\xi_1, \dots, \xi_m\rangle_k = S_{m+k}^+(\xi_m) S_{m-1+k}^+(\xi_{m-1}) \dots S_{2+k}^+(\xi_2) S_{1+k}^+(\xi_1) |0\rangle \quad (3.5)$$

with arbitrary k, m and ξ_1, \dots, ξ_m . The convenience of this definition comes from the fact that, due to the relations (3.1), the vectors (3.5) are symmetric with respect to all permutations of numbers ξ_1, \dots, ξ_m (exactly as in the Gaudin case). We denote the representation space of algebra (3.1) by $W_{F(\lambda), G(\lambda)}$.

4 The quasi-Gaudin model

Now we are ready to construct the integrable models associated with algebra (3.1). We postulate that Schrödinger equations for these models read

$$C_n(\lambda)\phi_n = E_n(\lambda)\phi_n, \quad \phi_n \in W_{F(\lambda), G(\lambda)}, \quad n = 0, 1, \dots \quad (4.1)$$

So, formula (4.1) defines an infinite sequence of integrable models, each of which is completely determined by a triple $\{F(\lambda), G(\lambda), n\}$.

5 A trivial solution

It is known that in the Gaudin case the lowest weight vector is always a solution of the Gaudin spectral problem. Is this true in the case of models (4.1)? In order to answer this question, we should simply act by the operator $C_n(\lambda)$ on the vector $|0\rangle$ and look at the result. For this it is convenient to rewrite the operator $C_n(\lambda)$ in a little bit different form. We use the formula

$$S_{n+1}^-(\lambda) S_n^+(\lambda) - S_{n-1}^+(\lambda) S_n^-(\lambda) = -2 \frac{\partial}{\partial \lambda} S_n^0(\lambda), \quad (5.1)$$

which follows from the quasi-commutation relations (3.1) in the limit $\mu \rightarrow \lambda$ and, substituting it into the expression (3.2) for $C_n(\lambda)$, obtain

$$C_n(\lambda) = S_n^0(\lambda) S_n^0(\lambda) + \frac{\partial}{\partial \lambda} S_n^0(\lambda) - S_{n-1}^+(\lambda) S_n^-(\lambda). \quad (5.2)$$

From (5.2) and formulas (3.4) it immediately follows that

$$C_n(\lambda)|0\rangle = \left\{ (F(\lambda) + nG(\lambda))^2 + \frac{\partial}{\partial \lambda} (F(\lambda) + nG(\lambda)) \right\} |0\rangle + nG(\lambda) S_{n-1}^+(\lambda) |0\rangle. \quad (5.3)$$

³Strictly speaking, the $G(\lambda)$ functions in the first and second formulas of (3.4) may differ from each other by a certain n -dependent constant factor. However, using the transformations $S_n^0(\lambda) \rightarrow S_n^0(\lambda)$, $S_n^-(\lambda) \rightarrow c_n S_n^-(\lambda)$, $S_n^+(\lambda) \rightarrow c_{n+1}^{-1} S_n^+(\lambda)$ which do not change the commutation relations (3.1), we can simply remove this difference.

Now it becomes clear that vector $|0\rangle$ could be a solution of problem (4.1) only if $n = 0$. In this case we obtain the relation

$$C_0(\lambda)|0\rangle = \left(F^2(\lambda) + \frac{\partial}{\partial\lambda}F(\lambda)\right)|0\rangle \quad (5.4)$$

which is similar to the Gaudin relation (2.7).

6 The Bethe ansatz

In analogy with the Gaudin case, let us try to find the solutions of spectral equations (4.1) by means of the algebraic Bethe ansatz.

It is natural to try to take the Bethe vector in the form

$$\phi_n = S_{m+k}^+(\xi_m)S_{m-1+k}^+(\xi_{m-1})\dots S_{2+k}^+(\xi_2)S_{k+1}^+(\xi_1)|0\rangle \quad (6.1)$$

with some k and m , and, using the relations

$$C_n(\lambda)S_{n-1}^+(\mu) - S_{n-1}^+(\mu)C_{n-1}(\lambda) = 2\frac{S_{n-1}^+(\mu)S_{n-1}^0(\lambda) - S_{n-1}^+(\lambda)S_{n-1}^0(\mu)}{\lambda - \mu}, \quad (6.2)$$

which follow from the basic relations (3.1), try to transfer the operator $C_n(\lambda)$ to the right. From formula (6.2) it is seen that, in order to start performing such a permutation, it is necessary to take

$$m + k = n - 1. \quad (6.3)$$

Now note that each permutation of the operator $C_n(\lambda)$ with raising generators decreases its index by one. This means that after m permutations when this operator appears in front of the lowest weight vector, it will have the index $n - m$. The standard prescriptions to Bethe ansatz technique imply that the hamiltonian of an integrable system, appearing after all permutations in front of the lowest weight vector, should be absorbed by it. But such an absorbion is possible only if the lowest weight vector is an eigenvector of a hamiltonian. According to the result of previous section, the lowest weight vector is an eigenvector of the operator $C_{n-m}(\lambda)$ only if

$$n - m = 0. \quad (6.4)$$

Comparing formulas (6.3) and 6.4 we can conclude that the only case when the ansatz (6.1) for equation (4.1) may lead to some algebraic solutions corresponds to the choice

$$k = -1, \quad m = n. \quad (6.5)$$

7 The Bethe ansatz solution

After using the restrictions (6.5), the ansatz (6.1) takes the form

$$\phi_n = S_{n-1}^+(\xi_n)S_{n-2}^+(\xi_{n-1})\dots S_1^+(\xi_2)S_0^+(\xi_1)|0\rangle. \quad (7.1)$$

Let us now check that it actually contains solutions of the problem (4.1). This can be demonstrated exactly in the same way as in the ordinary Gaudin case.

Repeating the reasonongs of ref. [17], we transfer the operator $C_n(\lambda)$ to the right by using the quasi-commutation relations (6.2). The neutral generators of quasi-Gaudin algebra appearing after such a transference also should be transfered to the right. Exactly as in the case of a C -operator, any permutation of a S^0 -operator with raising generators forming the Bethe vector decreases its index by one. Finally, after completing all possible transferences, we obtain a lot of terms with operators $C_0(\lambda)$, $S_0^0(\lambda)$ and $S_0^0(\xi_i)$ standing in front of the lowest weight vector $|0\rangle$. After this one should get rid of these operators by using formulas (3.4) and (5.4). The result has the form

$$C_n(\lambda)S_{n-1}^+(\xi_n)\dots S_0^+(\xi_1)|0\rangle = A(\lambda, \xi_1, \dots, \xi_n)S_{n-1}^+(\xi_n)\dots S_0^+(\xi_1)|0\rangle + \sum_{i=1}^n \frac{B_i(\xi_1, \dots, \xi_n)}{\lambda - \xi_i} S_{n-1}^+(\xi_n)\dots S_i^+(\xi_{i+1})S_{i-1}^+(\lambda)S_{i-2}^+(\xi_{i-1})\dots S_0^+(\xi_1)|0\rangle, \quad (7.2)$$

where $A(\lambda, \xi_1, \dots, \xi_n)$ and $B_i(\xi_1, \dots, \xi_n)$, $i = 1, \dots, n$ are some explicitly constructable functions. We see that the right hand side of (7.2) consists of two parts. The first part is proportional to the Bethe vector ϕ_n , while the second one consists of n terms which do not have such a form (these are the so-called "unwanted terms"). This means that the coefficient function $A(\lambda, \xi_1, \dots, \xi_n)$ determine the eigenvalues of the operator $C_n(\lambda)$ provided that the unwanted terms are absent. But in order to get rid of the unwanted terms it is sufficient to require the vanishing of their coefficients $B_i(\xi_1, \dots, \xi_n) = 0$, $i = 1, \dots, n$. This leads us to a system of n equations for n unknowns ξ_1, \dots, ξ_n , which is nothing else than a typical system of Bethe ansatz equations. The explicit form of these equations reads

$$\sum_{k=1, k \neq i}^n \frac{1}{\xi_i - \xi_k} + F(\xi_i) = 0, \quad i = 1, \dots, n, \quad (7.3)$$

and the final expression for the eigenvalues $E_n(\lambda)$ is given by the formula

$$E_n(\lambda) = F^2(\lambda) + \frac{\partial}{\partial \lambda} F(\lambda) + \sum_{i=1}^n \frac{F(\lambda) - F(\xi_i)}{\lambda - \xi_i} \quad (7.4)$$

Formulas (7.1), (7.3) and (7.4) complete the solution of problem (4.1). It is not difficult to see that this is only a partial solution since the linear hull of vectors (7.1) with fixed n gives us only a certain negligibly small part of the whole representation space defined in section 3. The above consideration enables one to conclude that the models which we obtained are typical quasi-exactly solvable models.

8 Simplest realizations of quasi-Gaudin algebra

In previous sections we did not discuss the realizations of algebra (3.1). Now it is a time to do this. In fact, we know only one realization which can be constructed from the generators of standard Gaudin algebra $\vec{S}(\lambda)$ satisfying the commutation relations (2.1). In order to construct it, we will need the special limiting operators

$$S^\pm = \lim_{\lambda \rightarrow \infty} \lambda S^\pm(\lambda), \quad S^0 = \lim_{\lambda \rightarrow \infty} \lambda S^0(\lambda), \quad (8.1)$$

which form the $sl(2)$ algebra. Using these operators, let us take

$$\begin{aligned} S_n^-(\lambda) &= S^-(\lambda) + \frac{n + f - S^0}{\lambda - a} \\ S_n^0(\lambda) &= S^0(\lambda) + \frac{n + f - S^0 + b}{\lambda - a} \\ S_n^+(\lambda) &= S^+(\lambda) + \frac{n + f - S^0 + 2b}{\lambda - a} \end{aligned} \quad (8.2)$$

where a and b are complex parameters and $f = \lim_{\lambda \rightarrow \infty} \lambda f(\lambda)$. It can be easily checked by direct calculations that operators (8.2) actually satisfy the quasi-commutation relations (3.1).

The lowest weight functions $F(\lambda)$ and $G(\lambda)$ characterizing the representation of this algebra read

$$F(\lambda) = f(\lambda) + \frac{b}{\lambda - a}, \quad G(\lambda) = \frac{1}{\lambda - a}. \quad (8.3)$$

Now it is absolutely clear that algebra (3.1) is a deformation of the Gaudin algebra. The role of the deformation parameter is played by number a . If $a \rightarrow \infty$, then the n -dependence of the operators (8.2) disappears and they transform into ordinary generators of Gaudin algebra. Respectively, the quasi-commutators in (3.1) become the ordinary ones. As to the formulas (8.3) defining the representations of algebra (4.1), they, in the limit $a \rightarrow \infty$ also transform into the relations (2.4) for Gaudin algebra.

9 Quasi-exact solvability: a different point of view

Substituting (8.2) into (3.2), we obtain the form of the operators (3.2):

$$\begin{aligned} C_n(\lambda) &= S^0(\lambda)S^0(\lambda) - \frac{1}{2}S^-(\lambda)S^+(\lambda) - \frac{1}{2}S^+(\lambda)S^-(\lambda) + \\ &+ \frac{2S^0(\lambda)(n + b + f - S^0) - S^-(\lambda)(n + 2b + f - S^0) - S^+(\lambda)(n + f - S^0)}{\lambda - a} + \\ &+ \frac{b(b - 1)}{(\lambda - a)^2} \end{aligned} \quad (9.1)$$

Let us now check in an independent way that the operators (9.1) actually describe quasi-exactly solvable models. First of all, note that if the formulas (8.2) hold, then the representation spaces of Gaudin and our algebras coincide. Denote by Φ_n the linear hull of all vectors $S^+(\xi_k) \dots S^+(\xi_1)|0\rangle$ with arbitrary ξ_1, \dots, ξ_k and $k \leq n$. It is known that if $f(\lambda)$ is a rational function, then $\dim \Phi_n < \infty$, for any n . From the obvious relations $S^\pm(\lambda)\Phi_n \subset \Phi_{n\pm 1}$, $S^0(\lambda)\Phi_n \subset \Phi_n$, $(n + f - S^0)\Phi_n \subset \Phi_{n-1}$ and $(n + f - S^0)\Phi_m \subset \Phi_m$ for $m \neq n$ it immediately follows that the operator $C_n(\lambda)$ admits only one algebraically constructable invariant subspace, Φ_n . This subspace is finite-dimensional and therefore the models (4.1) are quasi-exactly solvable.

10 Conclusion. From quasi- $G[sl(2)]$ to quasi- $sl(2)$

It is known that the $sl(2)$ algebra can be considered as a limiting case of the $sl(2)$ -Gaudin algebra (2.1). The generators of the latter are given by the formula (8.1).

It is natural to ask what kind of algebra will appear if we consider an analogous limit of the deformed algebra (3.1)? In analogy with the Gaudin case, we can define the generators of this algebra by formulas

$$S_n^\pm = \lim_{\lambda \rightarrow \infty} \lambda S_n^\pm(\lambda), \quad S_n^0 = \lim_{\lambda \rightarrow \infty} \lambda S_n^0(\lambda). \quad (10.1)$$

Multiplying the quasi-commutation relations (3.1) by $\lambda\mu$ and tending both λ and μ to infinity we easily derive the relations between these generators, which read

$$\begin{aligned} S_n^0 S_n^0 - S_n^0 S_n^0 &= 0, \quad S_{n\pm 1}^\pm S_n^\pm - S_{n\pm 1}^\pm S_n^\pm = 0, \\ S_{n\pm 1}^0 S_n^\pm - S_n^\pm S_{n\pm 1}^0 &= \pm S_n^\pm, \quad S_{n+1}^- S_n^+ - S_{n-1}^+ S_n^- = 2S_n^0. \end{aligned} \quad (10.2)$$

We can consider (10.2) as defining relations of a certain modification of the $sl(2)$ algebra which can be called “quasi- $sl(2)$ algebra”. Despite the fact that it *is not* a Lie algebra, it has many properties similar to those of the ordinary $sl(2)$. For example, it has a quasi-analogue of the Casimir operator,

$$C_n = S_n^0 S_n^0 - \frac{1}{2} S_{n+1}^- S_n^+ - \frac{1}{2} S_{n-1}^+ S_n^- \quad (10.3)$$

which quasi-commutes with all generators:

$$C_n S_{n-1}^+ = S_{n-1}^+ C_{n-1}, \quad C_n S_{n+1}^- = S_{n+1}^- C_{n+1}, \quad C_n S_n^0 = S_n^0 C_n. \quad (10.4)$$

The representations of quasi- $sl(2)$ algebra can be constructed in the same way as in quasi-Gaudin case. Defining the lowest weight vector $|0\rangle$ by the formulas

$$S_n^0 |0\rangle = (F + nG) |0\rangle, \quad S_n^- |0\rangle = nG |0\rangle, \quad (10.5)$$

where F and G are certain arbitrarily fixed numbers, we can define the representation space $W_{F,G}$ as linear hull of vectors

$$|m\rangle_k = S_{m+k}^+ S_{m-1+k}^+ \cdots S_{2+k}^+ S_{1+k}^+ |0\rangle, \quad (10.6)$$

with arbitrary k and m . If we are interested in the spectrum of quasi-Casimir operator, then it is easy to see that, in contrast with the standard $sl(2)$ case, it is not infinitely-degenerate and contains only a few number of exactly constructable eigenvectors. For example, the only possibility for vector (10.6) to be an eigenvector of (10.4) is realized when $k = -1$ and $m = n$. In this case, the corresponding eigenvalue is equal to $F(F-1)$. So, one can say that the quasi-Casimir operators represent the simplest (and, in some sense, trivial) quasi-exactly solvable models.

It is remarkable that the generators of quasi- $sl(2)$ algebra can be realized in terms of the generators of ordinary $sl(2)$ algebra. The corresponding formulas can be obtained after substitution of formulas (8.2) into (8.1) and read

$$S_n^- = S^- + n + f - S^0, \quad S_n^0 = n + b + f, \quad S_n^+ = S^+ + n + 2b + f - S^0. \quad (10.7)$$

This realization corresponds to the choice $F = f + b$ and $G = 1$ in (10.5). We do not know at the moment if there are other, more general, realizations of quasi- $sl(2)$ algebra.

Concluding this paper, let us mention two problems which arise as immediate consequences of the results obtained above. These are: 1) construction of general quasi-Gaudin algebras and models and 2) construction of general quasi-Lie algebras. However, one should stress again that our main goal is to develop a theory of general quasi-YB algebras. The first step in this direction will be done in our next publication which will appear soon in hep-th archive and in which we intend to construct quasi-exactly solvable deformations of the inhomogeneous XXX spin chain and investigate the associated quasi-Yangian $\mathcal{Y}[sl(2)]$.

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